Finite Element Exterior Calculus

A very incomplete introduction

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PhD Seminar

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Motivation

We want to solve (more or less)

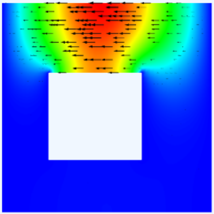
We want to solve equations like

$$(\mathrm{d}^* \mathrm{d} + \mathrm{d}\,\mathrm{d}^*)u = f$$

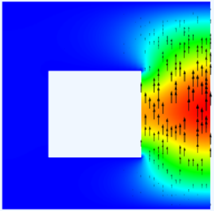
using mixed finite elements for

$$(u, \sigma = \mathrm{d}\, u).$$

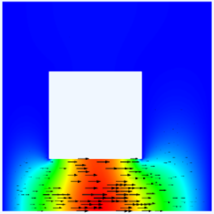
$\Delta u = \lambda u$, STRONG FORMULATION



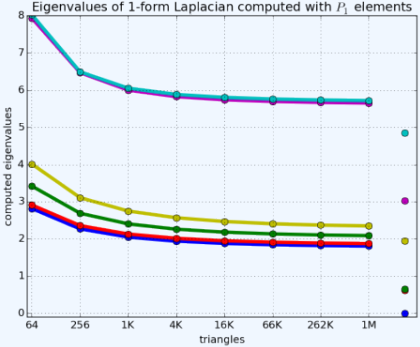
$\lambda_1 = 1.94$



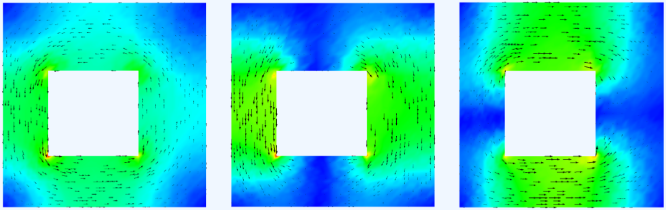
$\lambda_2 = 2.02$



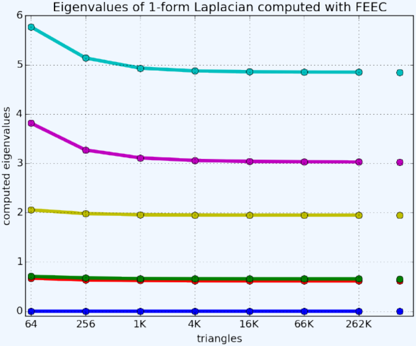
$\lambda_3 = 2.26$



$\Delta u = \lambda u$, MIXED FORMULATION



$\lambda_1 = 0$ $\lambda_2 = 0.617$ $\lambda_3 = 0.658$



Central is a short subchain of a Hilbert complex

$$\begin{array}{ccccc} V^{k-1} & \xrightarrow{\mathrm{d}} & V^{k} & \xrightarrow{\mathrm{d}} & V^{k+1} \\ \downarrow \pi_{k-1} & & \downarrow \pi_{k} & & \downarrow \pi_{k+1} \\ V_h^{k-1} & \xrightarrow{\mathrm{d}} & V_h^{k} & \xrightarrow{\mathrm{d}} & V_h^{k+1} \end{array}$$

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Three important properties for consistency and stability!

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- bounded projection property: π_h^l is bounded.

Contents

Abstract Hilbert complexes

... is a sequence of Hilbert spaces W^k and linear operators[1] d^k

$$\dots \xrightarrow{\mathrm{d}^{k-2}} W^{k-1} \xrightarrow{\mathrm{d}^{k-1}} W^k \xrightarrow{\mathrm{d}^k} W^{k+1} \xrightarrow{\mathrm{d}^{k+1}} \dots$$

such that

$$\mathrm{Im}\left(d^{k-1}\right) \subseteq \mathrm{Ker}\left(d^k\right).$$

[1]unbounded, closed, densely defined

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- Important property: $\mathrm{d} \circ \mathrm{d} = 0$.
- There is a norm $\|\cdot\|_V$, s.t. $d^{k-1} : V^{k-1} \to V^k$ is bounded.

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Recall: For a vector space W, the dual space is

$$W^* := \{\omega : V \to \mathbb{R} \mid \omega \text{ is linear and bounded.}\}.$$

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For

$$V \xrightarrow{A} W$$

we get the adjoint map

$$V^* \xleftarrow{A^*} W^*$$

via

$$A^*(\omega) : V \to \mathbb{R} : v \mapsto \omega(A(v)).$$

Turing around arrows is fun[2]

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{k-2}} & W^{k-1} & \xrightarrow{d^{k-1}} & W^{k} & \xrightarrow{d^{k}} & W^{k+1} & \xrightarrow{d^{k+1}} & \cdots \\
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$$

Be careful: The adjoint of an unbounded operator has a different domain!

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Abstract Hodge Laplace operator

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The harmonic forms

$$\operatorname{Ker}\left(L^k\right) = \operatorname{Ker}\left(d^k\right) \cap \operatorname{Ker}\left(d_{k-1}\right)$$

turn out to be crucial.

Never forget the kernel! We need to ensure existence of solutions

$$L^k u = f - \mathrm{Pr}_{\mathrm{Ker}\left(L^k\right)}(f)$$

and uniqueness

$$u \perp \mathrm{Ker}\left(L^k\right).$$

The Hodge Laplace equation

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Are there any interesting examples?

Three numbers $\neq$ vectors $\neq$ co-vectors

Three numbers

$$(1, 0, 0)$$

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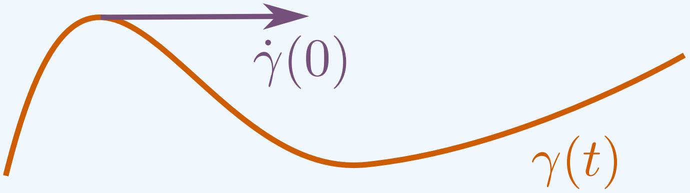
might represent a direction

$$\dot{\gamma} \in T\mathbb{R}^3$$



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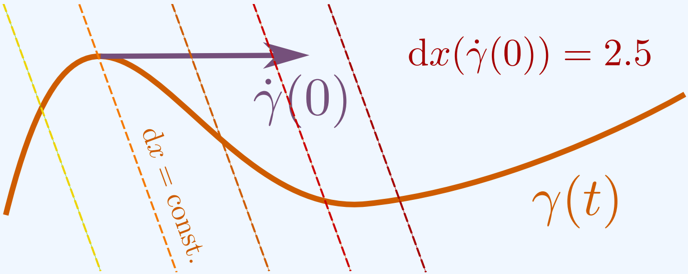

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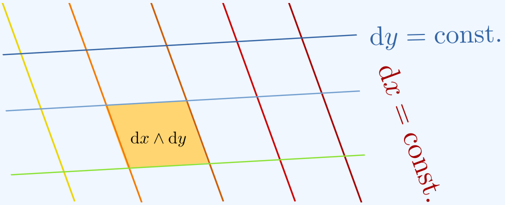
or an area element

$$\mathrm{d}\,x \wedge \mathrm{d}\,y \in \Lambda^2\mathbb{R}^3.$$



We define

$$\mathrm{Alt}^k(V) = \{\omega : V^k \to \mathbb{R} \mid \omega \text{ is linear and alternating}\}.$$

[3] $\phi : \mathbb{R}^n \to \mathbb{R}$, i.e. $\phi(p) \in \mathrm{Alt}^0(\mathbb{R}^n)$

Working definition of alternating forms

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Examples[3]

$$\mathrm{D}\,\phi(p) \in \mathrm{Alt}^1(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \qquad \det \in \mathrm{Alt}^n(\mathbb{R}^n).$$

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$$\mathrm{Alt}^k(V) \approx \text{ things that measure } k\text{-dimensional objects.}$$

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Working definition of differential forms

A smooth field of these 'measuring devices' is called a differential form

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We can define a derivative on these spaces via

$$\mathrm{d}\,\omega := \mathrm{skew}\,(\mathrm{D}\,\omega)\,, \quad \omega \in \Lambda^k(\Omega).$$

The de Rham (co-chain) complex

$$C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega)$$

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The maps between the rows are not trivial!

The L^2-de Rham (chain) complex

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L^2(\Omega) & \xrightarrow{(\mathrm{grad}, H^1)} & L^2(\Omega, \mathbb{R}^3) & \xrightarrow{(\mathrm{curl}, H(\mathrm{curl}))} & L^2(\Omega, \mathbb{R}^3) & \xrightarrow{(\mathrm{div}, H(\mathrm{div}))} & L^2(\Omega) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
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- $L^3 = -\operatorname{div}\operatorname{grad}$ + Dirichlet BC.

The cohomology spaces

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- If $\mathcal{H}^k = \{0\}$, then we find a 'potential'

$$\mathrm{d}\,\sigma = 0 \quad \Rightarrow \quad \sigma = \mathrm{d}\,u$$

  for some $u \in V^{k-1}$.

The cohomology spaces are a topological invariants!

Why do we care?

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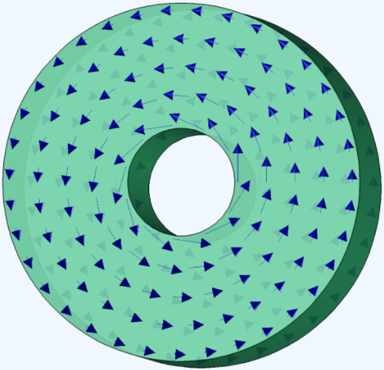
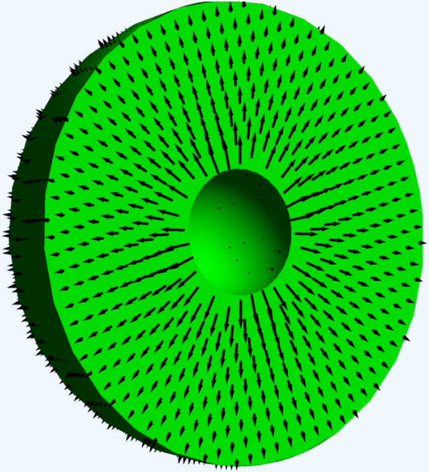


$u = \operatorname{grad} \theta, 0 \neq \bar{u} \in \mathcal{H}^1$
on cylindrical shell



$u = \operatorname{grad} \frac{1}{r}, 0 \neq \bar{u} \in \mathcal{H}^2$
on spherical shell

WHY DO WE CARE?

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Central philosophy of FFEC:
Try to preserve geometric invariants!

Example for Hodge Laplace equations and relatives

Using the subcomplex

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Elasticity

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$$H^{s-1}(\Omega) \otimes \mathbb{V} \xrightarrow{\text{grad}} H^{s-2}(\Omega) \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{s-2}(\Omega) \otimes \mathbb{S} \xrightarrow{\text{div}} H^{s-2}(\Omega) \otimes \mathbb{V}$$

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- $\text{inc}(F) := \underline{\text{curl}}\left(\left(\underline{\text{curl}}(F)\right)^T\right)$
- Mixed formulation of this scary complex are mixed elements for a (displacement, deformation, strain) formulation with strong symmetry.

Hodge wave equation

This beautiful equation

$$\begin{pmatrix} \dot{\sigma} \\ \dot{v} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d} & 0 \\ -\mathrm{d} & 0 & -\mathrm{d} \\ 0 & \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ v \\ \beta \end{pmatrix} + \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}$$

can be used to study

$$\begin{aligned} \dot{D} - \operatorname{curl} H &= -j, \\ \dot{B} + \operatorname{curl} E &= 0, \\ \operatorname{div} B &= 0, \\ \operatorname{div} D &= q. \end{aligned}$$

Discretisation of Hilbert complexes

For finite dimensional approximation spaces $V_h^l \subseteq V^l$, we can consider the induced Hilbert complex

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The essence of FEEC

For finite dimensional approximation spaces $V_h^l \subseteq V^l$, we can consider the induced Hilbert complex

$$\begin{array}{ccccc} V^{k-1} & \xrightarrow{\mathrm{d}} & V^k & \xrightarrow{\mathrm{d}} & V^{k+1} \\ \downarrow{\scriptstyle \pi_{k-1}} & & \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{k+1}} \\ V_h^{k-1} & \xrightarrow{\mathrm{d}} & V_h^k & \xrightarrow{\mathrm{d}} & V_h^{k+1} \end{array}$$

Three important properties for consistency and stability!

- approximation property: $\operatorname{dist}(V_h^l, w) \to 0$ for all $w \in V^l$
- subcomplex property: $\mathrm{d}\, V^l \subseteq V^{l+1}$
- bounded projection property: π_h^l is bounded.

Cohomology is preserved

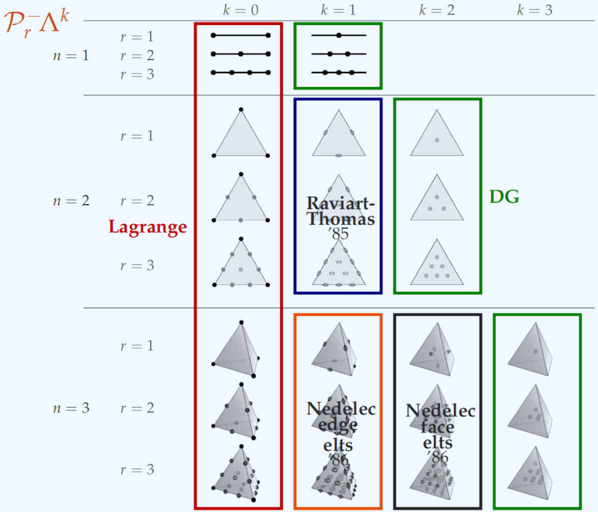
Under very mild conditions we get

$$\mathcal{H}_h^k \cong \mathcal{H}^k.$$

Typical tools from Sobolev theory also pop-up in the more general case of Hilbert complexes

$$\|z\| \leq c_P \|\mathrm{d}\, z\| \quad \text{for all } z \in \left(\mathrm{Ker}\left(d^k\right)\right)^{\perp_V}.$$

Periodic table of finite elements



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Still... very abstract and damm confusing.

Thanks for your **attention**!